J Glob Optim (2009) 43:391–405
DOI 10.1007/s10898-007-9223-4

Some transformation techniques with applications in global optimization

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Received: 10 April 2007 / Accepted: 28 June 2007 / Published online: 28 August 2007

Abstract In this paper some transformation techniques, based on power transformations, are discussed. The techniques can be applied to solve optimization problems including signomial functions to global optimality. Signomial terms can always be convexified and underestimated using power transformations on the individual variables in the terms. However, often not all variables need to be transformed. A method for minimizing the number of original variables involved in the transformations is, therefore, presented. In order to illustrate how the given method can be integrated into the transformation framework, some mixed integer optimization problems including signomial functions are finally solved to global optimality using the given techniques.

Keywords Transformation and convexification techniques · Signomial functions · Global optimization · Mixed integer non-linear programming

1 Introduction

The transformation techniques presented in this paper are especially useful when solving optimization problems including constraints composed of a convex and a signomial function. Such constraints can be convexified and underestimated by applying power transformations to the individual variables in the signomial terms, as long as, the inverse transformations of the power transformations are approximated with piecewise linear functions. The underestimation property is important when developing global optimization approaches, since it ensures that the feasible region of the convexified problem overestimates that of the original one. When the transformations are applied, the original non-convex feasible region is convexified and overestimated, as well as, divided into convex sub-regions by the variables used in the piecewise linear approximations of the inverse transformations. These transformation methods are studied in, for example [2,7].

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The fact that often not all variables need to be transformed in a term to be convexified, gives some degree of freedom regarding how the transformations can be chosen. Thus, to allow for a more efficient solution of the problem, the amount of transformations should be kept to a minimum, for which a method, of determining the optimal set of transformations convexifying the problem, is presented.

With the given techniques, some general classes of non-convex MINLP (mixed integer non-linear programming) problems can be solved to global optimality. The globally optimal solution to the original non-convex problem can be found by solving a sequence of convexified MINLP sub-problems. After each iteration, a part of the infeasible region is cut off when the piecewise linear approximations are updated. The algorithm terminates when a solution point is sufficiently close to, or within, the feasible region of the original problem.

2 The MINLP problem

A non-convex MINLP problem, which can be solved to global optimality by the transformation method can be written as

$$\begin{aligned} \min \quad & f(\mathbf{z}), && \mathbf{z} = (z_1, z_2, \ldots, z_I), \\ \text{s.t.} \quad & \mathbf{Az} = \mathbf{a}, \quad \mathbf{Bz} \le \mathbf{b}, && \\ & g_n(\mathbf{z}) \le 0, && n = 1, 2, \ldots, J_n, \\ & q_m(\mathbf{z}) + \sigma_m(\mathbf{z}) \le 0, && m = 1, 2, \ldots, J_m. \end{aligned} \tag{1}$$

Different methods to solve the convexified MINLP sub-problems can be applied. If the method used is the extended cutting plane (ECP) method [10], the objective function f in (1) can be differentiable pseudo-convex, and g, q and σ pseudo-convex, convex and signomial functions respectively. The vector $\mathbf{z}$ can consist of both continuous variables in a compact subset of a finite dimensional Euclidian space, as well as, integer variables in a finite dimensional integer set. The matrices $\mathbf{A}$ and $\mathbf{B}$, as well as, the vectors $\mathbf{a}$ and $\mathbf{b}$ should consist of constants only, and be of appropriate dimensions.

3 The convexity of signomial functions

The *signomial functions*, σ, in problem (1) are defined as the sum of signomial terms, where each term is a product of power functions, i.e.,

$$\sigma_m(\mathbf{z}) = \sum_{j=1}^{J} c_j \prod_{i=1}^{I} z_i^{p_{ji}}, \tag{2}$$

where $c_j, p_{ji} \in \mathbb{R}$. Obviously, a signomial function, σ, is convex if all the terms are convex. From now on, it is assumed that only non-convex signomial terms are included in the function, since it is always possible to move the convex signomial terms into the convex function q in (1). There are several different techniques for transforming non-convex signomial terms, such as the exponential transformation [6] and the inverse transformation [7]. The inverse transformation can be regarded as a special case of the transformation techniques, based on power transformations [9], used here. Convexity requirements for signomial terms are given by the following Theorem from [5]:

Theorem 1 *A positive signomial term $s(\mathbf{z}) = c\prod_{i=1}^{I} z_i^{p_i}$ is convex if one of the following statements is true*

(i) $p_i \leq 0, \quad \forall i = 1, \ldots, I,$
(ii) $\exists k: \ p_k + \sum_{i\neq k} p_i \geq 1$, *where* $p_i \leq 0, \ \forall i = 1, \ldots, I : i \neq k,$

and a negative signomial term $s(z) = c\prod_{i=1}^{I} z_i^{p_i}$ is convex if $p_i \geq 0, \ \forall i = 1, \ldots, I$, and $\sum_{i=1}^{I} p_i \leq 1$.

From Theorem 1 it can be deduced that it is always possible to convexify signomial terms using power transformations, and this fact will be used in the following chapter to convexify non-convex signomial terms.

4 The transformation approach

Using power transformations of the form $z = Z^Q$ to transform the signomial terms does convexify the terms for certain values of the powers Q, but only by moving the non-convexities from the signomial terms to the constraints introduced by the power transformations. However, by approximating the inverse power transformations, $Z = z^{1/Q}$, with piecewise linear functions, the whole problem can be convexified on the condition that the approximation of each transformed signomial term underestimates the original one. This can be guaranteed by introducing certain restrictions on the transformations. The transformation methods, as well as, the underestimation properties used in this paper, have been previously presented in, for example, [2,3,7,9]. Since the convexity requirements are different for positive and negative signomial terms, the power transformations applied to the individual variables must also fulfill different conditions depending on the sign of the term.

4.1 Convexifying and underestimating negative terms

According to Theorem 1, a negative signomial term is convex if all the powers are positive and their sum is less than or equal to one. By using the power transformation

$$z_i = Z_i^{Q_i} \Rightarrow Z_i = z_i^{1/Q_i}$$

on each original variable z_i included in the term, with the following conditions on the powers Q

$$\begin{cases} Q_i > 0, & \text{if } p_i > 0, \\ Q_i < 0, & \text{if } p_i < 0, \\ Q_i = 0, & \text{if } p_i = 0, \end{cases}$$

the term will be convexified, as long as the following inequality is true:

$$\sum_{i=1}^{I} p_i Q_i \leq 1. \tag{3}$$

Note that each term $p_i Q_i$ in (3) should be positive. Thus, the sum in the inequality is always larger than zero for negative signomial terms and the inequality can always be satisfied if Q_i is chosen close enough to zero from either the positive or the negative side. Applying the power transformations to the individual variables with the powers Q defined as above, yields the following expression for the transformed signomial term

$$c \prod_{i=1}^{I} Z_i^{p_i Q_i}. \tag{4}$$

For the signomial term to be underestimated when approximating the individual power transformations with piecewise linear functions, the powers Q have to fulfill certain criteria [9], and when combining these with the convexification conditions, the following requirements are received for the signomial term to be convexified and underestimated:

$$\begin{cases} 0 < Q_i \leq 1, & \text{if } p_i > 0, \\ Q_i < 0, & \text{if } p_i < 0, \end{cases} \quad \text{and} \quad \sum_{i=1}^{I} p_i Q_i \leq 1. \tag{5}$$

4.2 Convexifying and underestimating positive terms

According to Theorem 1, a positive signomial term is convex if all powers are negative, or at most one power is positive and the sum of the powers is greater than or equal to one. Using the power transformation

$$z_i = Z_i^{Q_i} \Rightarrow Z_i = z_i^{1/Q_i}$$

on each original variable z_i included in the term, with the following conditions on the powers Q

$$\begin{cases} Q_i > 0, & \text{if } p_i > 0 \wedge i = k, \\ Q_i < 0, & \text{if } p_i > 0 \wedge i \neq k, \\ Q_i = 1, & \text{if } p_i < 0. \end{cases}$$

will convexify the term. The index $k : 1 \leq k \leq I$, which may or may not exist, corresponds to the power remaining positive after the transformation, i.e., the product $p_k Q_k$ should be positive, while the products $p_i Q_i$ should be negative for all other indices $i \neq k$. If a power is positive after the transformation, the sum of the powers has to be larger or equal to one. Thus, in this case, the following condition must also be fulfilled

$$\sum_{i=1}^{I} p_i Q_i \geq 1.$$

Furthermore, since a variable with a negative power does not need to be transformed, the power Q is equal to one for all the negative powers. After the transformation, the signomial term will look like expression (4). As in the case with the negative signomial term, the powers Q must, furthermore, fulfill certain criteria for the signomial term to be underestimated [9]. The combination of these criteria with the convexification requirements yields the following conditions on the powers Q:

$$\begin{cases} Q_i \geq 1, & \text{if } p_i > 0 \wedge i = k, \\ Q_i < 0, & \text{if } p_i > 0 \wedge i \neq k, \\ Q_i = 1, & \text{if } p_i < 0. \end{cases}$$

Using power transformations of the mentioned form ensures that it is always possible to convexify non-convex signomial terms. The transformed term is also underestimated whenever the inverse power transformations are approximated by piecewise linear functions.

4.3 Piecewise linear functions using special ordered sets

One way to model piecewise linear functions is using so-called *special ordered sets* (SOS). This method is usually computationally more efficient than methods using binary variables. A special ordered set of type 2 is a set of variables (integers, continuous or mixed integer and continuous), where at most two variables may be non-zero, and if two variables are non-zero, they must be adjacent in the set.

The following method of approximating piecewise linear functions using SOS is from [1]: A piecewise linear approximation of the inverse power transformations, $Z = z^{1/Q}$, with the values $Z_k = z_k^{1/Q}$ at the breakpoints z_k, for $k = 1, \ldots, K$, can be written as

$$\hat{Z} = \sum_{k=1}^{K} Z_k \cdot w_k, \quad w_k \geq 0,$$

where

$$z = \sum_{k=1}^{K} z_k \cdot w_k, \qquad \sum_{k=1}^{K} w_k = 1,$$

and $\{w_k\}_{i=1}^{K}$ is a special ordered set of type 2 with the weights $\{z_k\}_{k=1}^{K}$. The weights are used in the MILP algorithm to order the variables, and therefore, all weights must have different values. Here, this requirement means that $z_k \neq z_j$ for all $k \neq j$. It should be noted that, when applying several different transformations $Z = z^{1/Q}$ to the same original variable z in different signomial terms, the same variables w_k can still be used in all the piecewise linear approximations of the different inverse transformations.

4.4 An example of a one-dimensional underestimation

All terms in the non-convex function

$$f(x) = (x^4 + 79.5x^2 - 170x + 120) - 15x^3, \quad 1 \leq x \leq 6,$$

are convex, except for $-15x^3$. Using the power transformation $x = X^Q$ with $Q = 1/3$ and approximating X with a piecewise linear function $\hat{X}$ taking values X_k at the breakpoints x_k, gives the following expression for the function $f(x, \hat{X})$:

$$f(x, \hat{X}) = (x^4 + 79.5x^2 - 170x + 120) - 15\hat{X}$$

$$\hat{X} = \sum_{k=1}^{K} X_k \cdot w_k, \qquad w_k \geq 0,$$

$$x = \sum_{k=1}^{K} x_k \cdot w_k, \qquad \sum_{k=1}^{K} w_k = 1.$$

The underestimations of the function $f(x)$ in $K = 1, 2, 4, 8$ equidistant steps are illustrated in Fig. 1.

5 Optimization of the transformations

The convexification and underestimation requirements mentioned above sometimes allow for the power Q to take the value one, indicating that no transformation occurs. A method for

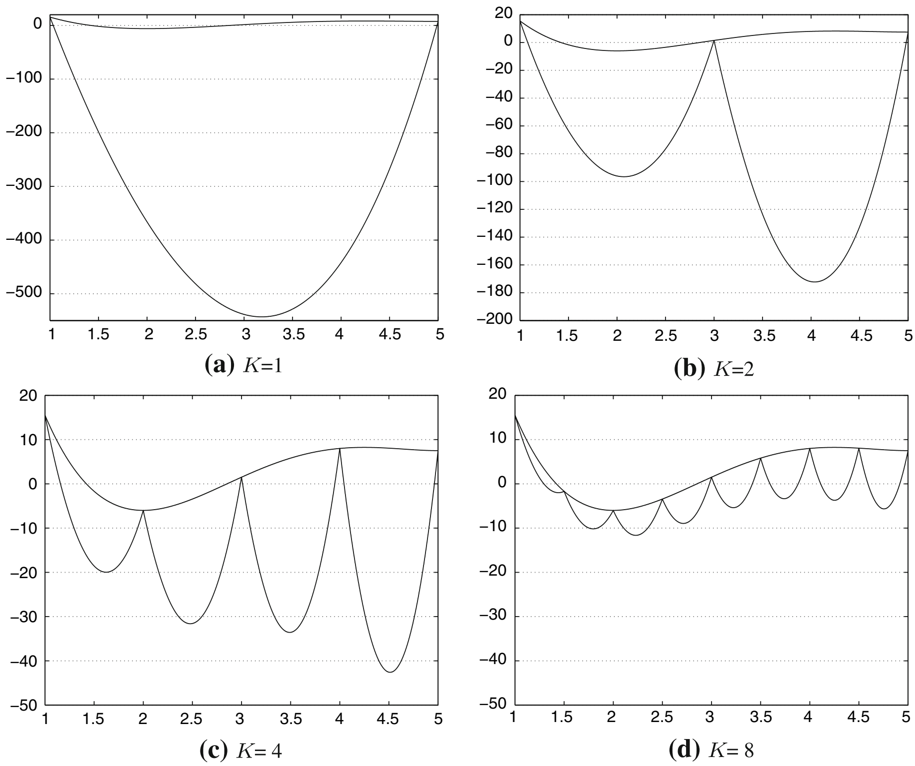


Fig. 1 The function $f(x) = (x^4 + 79.5x^2 - 170x + 120) - 15x^3$ and the convex underestimations $f(x, \hat{X})$ in $K = 1, 2, 4, 8$ steps (**a**) $K = 1$ (**b**) $K = 2$ (**c**) $K = 4$ (**d**) $K = 8$

optimizing the transformation approach [4], is described in this chapter, making it possible to determine which variables need to be transformed, as well as, what values of the powers should be used in the transformations.

The index j now corresponds the j-th non-convex signomial term in the MINLP problem, which has a total of J_T non-convex signomial terms. By introducing a binary variable b_{ji} taking the value one if the i-th variable in the j-th signomial term is transformed by a power transformation, and zero otherwise, the transformed signomial term can be written as

$$c_j \prod_{i=1}^{I} z_i^{(1-b_{ji})p_{ji}} \cdot Z_{ji}^{b_{ji}p_{ji}Q_{ji}},$$

since $z^{(1-b)p} \cdot Z^{bpQ}$ simplifies to z^p whenever b is zero and to Z^{pQ} if b is one. A mixed integer linear programming (MILP) problem with the objective being to minimize the number of transformations required to convexify and underestimate the non-convex signomial terms can then be formulated. The total number of transformations required are

$$\sum_{j=1}^{J_T} \sum_{i=1}^{I} b_{ji}. \tag{6}$$

However, as mentioned previously, when approximating the inverse transformations with piecewise linear functions, the same variables can be used in the different linear approximations, even if the power transformations used are not the same. Therefore, it is of greater importance to minimize the total number of original variables transformed and thus minimizing the number of variables needed for the piecewise linear approximations of the inverse transform. This is accomplished by introducing a new binary variable B_i, for each of the original variables included in the signomial terms, being equal to one if the i-th variable is involved in any transformation and zero otherwise. This can be expressed as

$$\sum_{j=1}^{J_T} b_{ji} \leq J_T B_i.$$

Minimizing the number of original variables transformed, but still also favoring solutions with less transformations, the sum (6), multiplied with a small positive number δ_1, is additionally included in the objective function. Furthermore, variables Δ corresponding to the deviation of the powers Q from $+1$ or -1, depending on whether Q is positive or negative are introduced. To promote powers Q closer to $+1$ or -1 an additional penalty term is included (multiplied with δ_2) in the objective function. Hence, the final objective function becomes:

$$\sum_{i=1}^{I} B_i + \delta_1 \sum_{j=1}^{J_T} \sum_{i=1}^{I} b_{ji} + \delta_2 \sum_{j=1}^{J_T} \sum_{i=1}^{I} \Delta_{ji}. \tag{7}$$

The requirements on the binaries B and the deviations Δ can be expressed as

$$\begin{cases} \sum_{j=1}^{J_T} b_{ji} \leq J_T B_i, \\ Q_{ji} - \Delta_{ji} + M\beta_{ji} \leq M + 1, \\ -Q_{ji} - \Delta_{ji} + M\beta_{ji} \leq M - 1, \end{cases} \quad \text{and} \quad \begin{cases} Q_{ji} - \Delta_{ji} - M\beta_{ji} \leq -1, \\ -Q_{ji} - \Delta_{ji} - M\beta_{ji} \leq 1, \\ M(\beta_{ji} - 1) \leq Q_{ji} \leq M\beta_{ji}, \end{cases} \tag{8}$$

where M is a large positive number and Δ is the deviation from +1 or -1 if Q is positive (when the binary $\beta = 1$) or negative (when the binary $\beta = 0$), respectively. The weights δ_1 and δ_2 can be given different values depending on what types of transformations are wanted; increasing the value of δ_1 promotes less transformations and the value of δ_2 results in numerically more stable transformations.

Conditions guaranteeing that the term is convex after transformation must also be added to the linear problem. Because the convexity requirements for positive and negative signomial terms are different, the formulations of these conditions are also different in the two cases.

5.1 Conditions for the negative signomial terms

For a positive power ($p > 0$) in a negative signomial term ($c < 0$), conditions must be included that ensure that whenever a transformation is necessary ($b = 1$) then Q must be between zero and one, and also, whenever a transformation is not needed ($b = 0$), then Q should be equal to one. Furthermore, for the convexified term to be convex, the sum of the products $p_{ji} Q_{ji}$ should be less or equal to one, so the following inequality must also be included:

$$\sum_{i=1}^{I} p_{ji} Q_{ji} \leq 1. \tag{9}$$

These requirements can be formulated by the following inequalities:

$$\begin{cases} Q_{ji} \geq 1 - b_{ji}, \\ Q_{ji} \leq 1 - \epsilon\, b_{ji}, \\ Q_{ji} \geq \epsilon, \end{cases} \Rightarrow \begin{cases} b_{ji} = 0 : Q_{ji} = 1, \\ b_{ji} = 1 : \epsilon \leq Q_{ji} \leq 1 - \epsilon. \end{cases} \tag{10}$$

A constant $\epsilon = 1/M$, where M is a large positive number, has been used to receive practical bounds on the powers Q.

For a negative power ($p < 0$) in a negative signomial term ($c < 0$), a transformation is always necessary and Q should be negative, resulting in the requirements:

$$b_{ji} = 1 \quad \text{and} \quad -M \leq Q_{ji} \leq -\epsilon. \tag{11}$$

5.2 Conditions for the positive signomial terms

In the case of a positive signomial term ($c > 0$), more freedom exists regarding how the transformations can be chosen, since there are two different ways to convexify the term according to Theorem 1: Either all variables have negative powers after the transformation, or one has a positive power, and the rest have negative powers. Furthermore, if a positive power exists after the transformation, the sum of the powers must be greater than or equal to one. Therefore, a binary variable α_{ji}, equal to one if z_i in the j-th signomial term has a positive power and equal to zero otherwise, is introduced, At most one variable per term is allowed to have a positive power after the transformation, so the following requirement must be included:

$$\sum_{i=1}^{I} \alpha_{ji} \leq 1. \tag{12}$$

For a variable with positive power after the transformation ($\alpha = 1$), the variable Q should be greater or equal to one, and smaller than zero for the rest of the variables ($\alpha = 0$). This can be expressed by the following inequalities:

$$\begin{cases} Q_{ji} \leq \alpha_{ji} M - \epsilon(1 - \alpha_{ji}), \\ Q_{ji} \geq -M + \alpha_{ji}(M + 1), \end{cases} \Rightarrow \begin{cases} \alpha_{ji} = 0 : \; -M \leq Q_{ji} \leq -\epsilon, \\ \alpha_{ji} = 1 : \; 1 \leq Q_{ji} \leq M. \end{cases} \tag{13}$$

Also, the binary b, indicating whether a transformation occurs or not, should be equal to zero when α and Q both are equal to one, and equal to one otherwise. These conditions can be formulated as:

$$\begin{cases} b_{ji} \geq 1 - \alpha_{ji}, \\ b_{ji} \geq \epsilon(Q_{ji} - 1), \\ b_{ji} \leq (1 - \epsilon) Q_{ji} + M(1 - \alpha_{ji}), \end{cases} \Rightarrow \begin{cases} \alpha_{ji} = 0 \wedge Q_{ji} < 0 : & b_{ji} = 1, \\ \alpha_{ji} = 1 \wedge Q_{ji} = 1 : & b_{ji} = 0, \\ \alpha_{ji} = 1 \wedge Q_{ji} \geq \frac{1}{1-\epsilon} : & b_{ji} = 1. \end{cases} \tag{14}$$

In these inequalities, the same values on M and ϵ can be used as in the case with the negative signomial terms. For a variable originally having a negative power ($p < 0$), no transformation is needed, so

$$Q_{ji} = 1, \; b_{ji} = 0 \quad \text{and} \quad \alpha_{ji} = 0. \tag{15}$$

The convexity condition for a positive signomial term is, according to Theorem 1, that if a variable has a positive power after the transformation (i.e., $\sum_{i=1}^{I} \alpha_{ji} = 1$) then the sum of the powers in the term should be greater than or equal to one, and otherwise less than zero.

Hence the following expression (where M is a large positive number) must be included for each positive signomial term:

$$\sum_{i=1}^{I} p_{ji} Q_{ji} - M \sum_{i=1}^{I} \alpha_{ji} \geq 1 - M. \tag{16}$$

Solving a MILP problem minimizing (7) subject to the linear requirements in (8–16) will indicate, not only how many transformations are required to convexify and underestimate the original MINLP problem (1) but also, which power transformations should be used.

5.3 Some numerical considerations

Since signomial terms (transformed or not transformed) can include power terms with both small and large powers, numerical difficulties may appear in the calculation of individual terms. A simple approach to overcome some numerical difficulties of this form is suggested here.

The variables being transformed are defined by the index set I_j. Now, observe that if variables are transformed in the j-th signomial term, the convexified signomial term underestimating the original term (i.e., $\hat{s}_j(\mathbf{z}) \leq s_j(\mathbf{z})$), can be written as

$$\hat{s}_j(\mathbf{z}) = c_j \prod_{i \in I_j} \hat{Z}_{ji}^{p_{ji} Q_{ji}} \cdot \prod_{i \notin I_j} z_i^{p_{ji}}, \tag{17}$$

where the first product corresponds to the estimated transformation variables, given by piecewise linear functions of the inverse transformations, and the second product to all other variables (not transformed). The piecewise linear approximations of the inverse transformations at K_i breakpoints $z_{i,k}$ of the variables z_i are in explicit form, according to Chapter 4.3, given by the expressions

$$\begin{cases} \hat{Z}_{ji} = \sum_{k=1}^{K_i} z_{i,k}^{1/Q_{ji}} \cdot w_{i,k}, \\ z_i = \sum_{k=1}^{K_i} z_{i,k} \cdot w_{i,k}, \quad \sum_{k=1}^{K_i} w_{i,k} = 1, \end{cases} \qquad \forall i \in I_j. \tag{18}$$

In order to overcome possible numerical difficulties arising from small and large powers in Eq. 17, an alternative is to rewrite the equation as follows:

$$\hat{s}_j(\mathbf{z}) = \exp\left[\ln(c_j) + \sum_{i \in I_j} (p_{ji} Q_{ji}) \cdot \ln(\hat{Z}_{ji}) + \sum_{i \notin I_j} p_{ji} \cdot \ln(z_i) \right]. \tag{19}$$

Using this approach, the j-th signomial term (17) can be expressed in the numerically more stable form (19).

6 The GGPECP algorithm

The GGPECP algorithm, described in [11], combines the transformation techniques presented in Chapters 4 and 5 with the extended cutting plane (ECP) algorithm from [10], to solve non-convex MINLP problems containing signomial functions to global optimality as a sequence of convexified sub-problems. The algorithm also utilizes information from previous iterations to make it possible to reach the solution more efficiently.

Before the first iteration, an initial transformation is performed, where the non-convex signomial terms are convexified using, for example, the power transformations mentioned earlier. By approximating and underestimating the inverse power transformation with piecewise linear functions, the feasible region of the convexified problem is overestimated. If the solution found by solving the convexified problem is feasible also in the original problem, it is the globally optimal solution, and the algorithm can be terminated. Otherwise, additional break points are added to the piecewise linear approximations of the inverse transformations, after which the new sub-problem is solved. In [11] different strategies for which grid points to add in each step are discussed. One of the strategies is to add the values of the variables from the solution of the previous iteration as new grid points, and this method is implemented in the following example.

6.1 A numerical example in two dimensions

The following is a two dimensional MINLP problem:

$$\begin{aligned} \min \quad & y - 3x, \\ \text{s.t.} \quad & y + 5x \leq 36, \quad -y + 0.25x \leq -1, \\ & (2y^2 - 2y^{0.5} + 11y + 8x - 39) - 2x^{0.5}y^2 + 0.1x^{1.5}y^{1.5} \leq 0, \\ & 1 \leq x \leq 7, \quad 1 \leq y \leq 7, \quad x \in \mathbb{R}^+, \quad y \in \mathbb{Z}^+. \end{aligned} \tag{20}$$

The integer-relaxed feasible region of this problem is divided into two disjoint regions as shown in Fig. 2. All inequalities, as well as, the objective function in this problem are linear, except for the generalized signomial constraint. This constraint is the sum of a convex function and a signomial function consisting of two non-convex signomial terms. Applying the method described in Chapter 5 (with the weights $\delta_1 = 0.01$ and $\delta_2 = 0.001$) to the non-convex terms gives the MILP problem in Appendix A. Solving the MILP problem gives that the variable x does not need to be transformed at all, while the variable y will be transformed in both terms according to:

$$\begin{aligned} y = Y_1^{0.25} \quad & \Rightarrow Y_1 = y^4, \\ y = Y_2^{-1/3} \quad & \Rightarrow Y_2 = y^{-3}. \end{aligned} \tag{21}$$

By replacing the variable y in the non-convex terms with Y_1 and Y_2, the generalized signomial constraint can be written in convex form as:

$$(2y^2 - 2y^{0.5} + 11y + 8x - 39) - 2x^{0.5}Y_1^{0.5} + 0.1x^{1.5}Y_2^{-0.5} \leq 0. \tag{22}$$

The inverse transformations $Y_1 = y^4$ and $Y_2 = y^{-3}$ are approximated by piecewise linear functions with initial breakpoints $y = 1$ and $y = 7$ according to:

$$\begin{aligned} \hat{Y}_1 &= 1 \cdot w_1 + 2401 \cdot w_2, \\ \hat{Y}_2 &= 1 \cdot w_1 + 0.0029 \cdot w_2, \\ y &= 1 \cdot w_1 + 7 \cdot w_2, \\ w_1 + w_2 &= 1, \end{aligned}$$

where w_1 and w_2 belong to a special ordered set of type 2. Using these approximations (i.e., replacing Y_1 and Y_2 with $\hat{Y}_1$ and $\hat{Y}_2$ in Eq. 22) the feasible region of the problem is

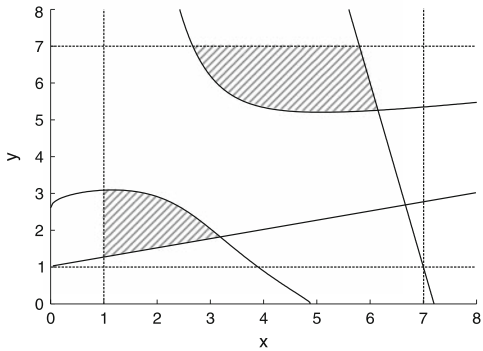


Fig. 2 The original integer-relaxed feasible region of the MINLP problem (20)

overestimated, which is illustrated in Fig. 3a. Solving the convexified MINLP problem gives the solution $x = 6.6$, $y = 3$, the objective value -16.8 and the value 23.9035 of the original generalized signomial constraint. By adding the solution point $y = 3$ as an additional grid point to the linear approximations in the next iteration, the approximation is improved:

$$\begin{aligned}
\hat{Y}_1 &= 1 \cdot w_1 + 81 \cdot w_2 + 2401 \cdot w_3,\\
\hat{Y}_2 &= 1 \cdot w_1 + 0.0370 \cdot w_2 + 0.0029 \cdot w_3,\\
y &= 1 \cdot w_1 + 3 \cdot w_2 + 7 \cdot w_3,\\
w_1 &+ w_2 + w_3 = 1.
\end{aligned}$$

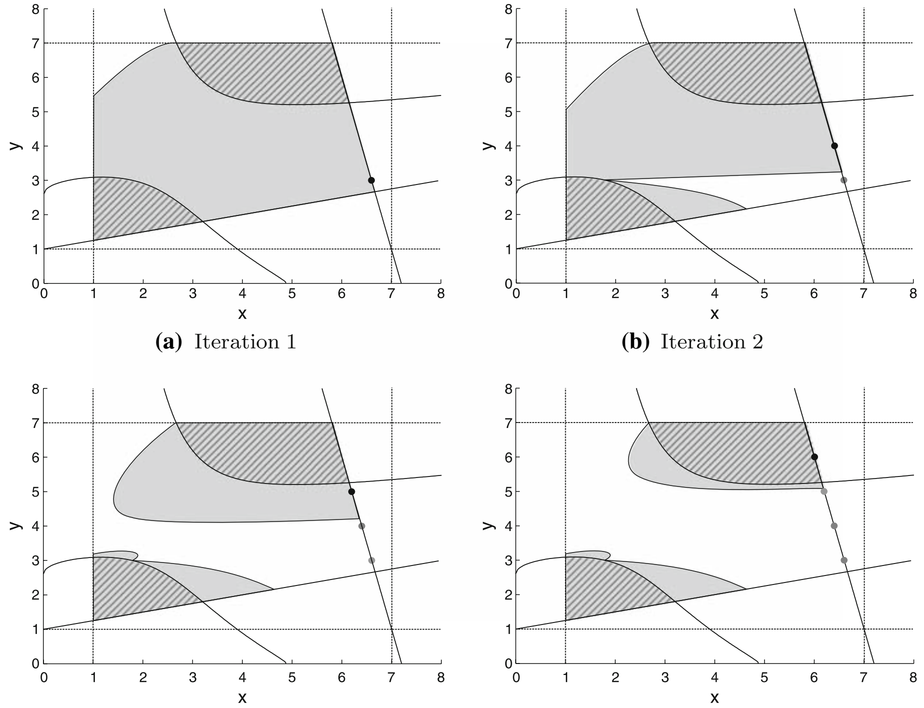
The feasible region of the problem at iteration 2 is shown in Fig. 3b. Note that the previous solution point is infeasible in the new feasible region.

Solving the problem again with the updated piecewise linear approximations gives the solution $x = 6.4$, $y = 4$, the objective value -15.2 and the value 16.1984 of the original signomial constraint. Since the signomial constraint is not yet satisfied, more iterations are needed. Adding the solution value $y = 4$ as a new grid point to the linear approximation gives:

$$\begin{aligned}
\hat{Y}_1 &= 1 \cdot w_1 + 81 \cdot w_2 + 256 \cdot w_3 + 2401 \cdot w_4,\\
\hat{Y}_2 &= 1 \cdot w_1 + 0.0370 \cdot w_2 + 0.0156 \cdot w_3 + 0.0029 \cdot w_4,\\
y &= 1 \cdot w_1 + 3 \cdot w_2 + 4 \cdot w_3 + 7 \cdot w_4,\\
w_1 &+ w_2 + w_3 + w_4 = 1.
\end{aligned}$$

The new feasible region is shown in Fig. 3c, and the solution found in this iteration is $x = 6.2$, $y = 5$, the objective value -13.6 and the value 3.8889 of the original signomial constraint. Again updating the grid points with the previous solution ($y = 5$) gives the following linear approximation:

$$\begin{aligned}
\hat{Y}_1 &= 1 \cdot w_1 + 81 \cdot w_2 + 256 \cdot w_2 + 625 \cdot w_4 + 2401 \cdot w_5,\\
\hat{Y}_2 &= 1 \cdot w_1 + 0.0370 \cdot w_2 + 0.0156 \cdot w_3 + 0.0080 \cdot w_4 + 0.0029 \cdot w_5,\\
y &= 1 \cdot w_1 + 3 \cdot w_2 + 4 \cdot w_3 + 5 \cdot w_4 + 7 \cdot w_5,\\
w_1 &+ w_2 + w_3 + w_4 + w_5 = 1.
\end{aligned}$$



(a) Iteration 1 **(b)** Iteration 2

(c) Iteration 3 **(d)** Iteration 4

Fig. 3 The overestimated integer-relaxed feasible region of the problem during the GGPECP iterations. The solution point at each iteration is marked in black

The final overestimated feasible region is shown in Fig. 3d. Solving this problem gives the solution $x = 6$ and $y = 6$ with the objective function value -12, which is globally optimal, since the value of the generalized signomial constraint is -12.6622 and the solution was obtained in an overestimated feasible region.

6.2 A geometric programming example

The following example is a geometric programming problem from [8], and includes a signomial objective function, as well as, seven signomial inequality constraints. In total eight variables are used.

$$
\begin{aligned}
\min \quad & 2.0\, z_1^{0.9} z_2^{-1.5} z_3^{-3} + \underline{5.0\, z_4^{-0.3} z_5^{2.6}} + 4.7\, z_6^{-1.8} z_7^{-0.5} z_8, \\
\text{s.t.} \quad & 7.2\, z_1^{-3.8} z_2^{2.2} z_3^{4.3} + \underline{0.5\, z_4^{-0.7} z_5^{-1.6}} + 0.2\, z_6^{4.3} z_7^{-1.9} z_8^{8.5} \le 1, \\
& 10.0\, z_1^{2.3} z_2^{1.7} z_3^{4.5} \le 1, \qquad 0.6\, z_4^{-2.1} z_5^{0.4} \le 1, \\
& \underline{6.2\, z_6^{4.5} z_7^{-2.7} z_8^{-0.6}} \le 1, \qquad 3.1\, z_1^{1.6} z_2^{0.4} z_3^{-3.8} \le 1, \\
& 3.7\, z_4^{5.4} z_5^{1.3} \le 1, \qquad 0.3\, z_6^{-1.1} z_7^{7.3} z_8^{-5.6} \le 1.
\end{aligned}
$$

Table 1 The original terms, the transformed terms, as well as, the obtained power transformations to transform the problem (23)

Original term	Transformed term	Transformations
$2.0\, z_1^{0.9} z_2^{-1.5} z_3^{-3}$	$2.0\, Z_{1,1}^{-0.9} z_2^{-1.5} z_3^{-3}$	$z_1 = Z_{1,1}^{-1}$
$4.7\, z_6^{-1.8} z_7^{-0.5} z_8$	$4.7\, z_6^{-1.8} z_7^{-0.5} Z_{2,8}^{-1}$	$z_8 = Z_{2,8}^{-1}$
$7.2\, z_1^{-3.8} z_2^{2.2} z_3^{4.3}$	$7.2\, z_1^{-3.8} Z_{3,2}^{-2.2} Z_{3,3}^{7}$	$z_2 = Z_{3,2}^{-1},\ z_3 = Z_{3,3}^{1.6279}$
$0.2\, z_6^{4.3} z_7^{-1.9} z_8^{8.5}$	$0.2\, z_6^{4.3} z_7^{-1.9} Z_{4,8}^{-1.4}$	$z_8 = Z_{4,8}^{-0.16471}$
$10.0\, z_1^{2.3} z_2^{1.7} z_3^{4.5}$	$10.0\, Z_{5,1}^{-1.8} Z_{5,2}^{-1.7} z_3^{4.5}$	$z_1 = Z_{5,1}^{-0.78261},\ z_2 = Z_{5,2}^{-1}$
$0.6\, z_4^{-2.1} z_5^{0.4}$	$0.6\, z_4^{-2.1} Z_{6,5}^{-0.4}$	$z_5 = Z_{6,5}^{-1}$
$3.1\, z_1^{1.6} z_2^{0.4} z_3^{-3.8}$	$3.1\, Z_{7,1}^{5.2} Z_{7,2}^{-0.4} z_3^{-3.8}$	$z_1 = Z_{7,1}^{3.25},\ z_2 = Z_{7,2}^{-1}$
$3.7\, z_4^{5.4} z_5^{1.3}$	$3.7\, z_4^{5.4} Z_{8,5}^{-1.3}$	$z_5 = Z_{8,5}^{-1}$
$0.3\, z_6^{-1.1} z_7^{7.3} z_8^{-5.6}$	$0.3\, z_6^{-1.1} Z_{9,7}^{-7.3} z_8^{-5.6}$	$z_7 = Z_{9,7}^{-1}$

The underlined terms are convex, and can therefore be included in the convex functions q in (1), whilst the rest of the terms are included in the signomial functions σ. Utilizing the method from Chapter 5 (with the weights $\delta_1 = 0.01$ and $\delta_2 = 0.0001$) to obtain the transformations, results in 12 power transformations in total being needed to convexify and underestimate the signomial terms, and that the variables z_4 and z_6 do not require any transformation at all. The resulting convexified terms and the power transformations needed for the transformations are given in Table 1. Solving the convexified and underestimated problem with the GGPECP algorithm gives the objective value 29.2291 and the following variable values:

$$z_1 = 0.9688,\ z_2 = 0.1990,\ z_3 = 1.1213,\ z_4 = 0.7844,$$
$$z_5 = 1.0022,\ z_6 = 0.7007,\ z_7 = 1.0934,\ z_8 = 0.9717.$$

This solution is slightly better than in [8], where the objective value was 29.5985.

7 Conclusions

Transformation techniques that can be used on optimization problems involving non-convex signomial terms were discussed in this paper. The techniques are based on power transformations in combination with piecewise linear approximations of the inverse power transformations. This makes it possible to convexify and underestimate signomial constraints and to solve problems including signomial functions to global optimality. An optimization method for obtaining the power transformations was presented, and applying this method in an initial convexification step in the GGPECP algorithm allows for the possibility to solve signomial problems more efficiently. In order to illustrate the transformation techniques, two problems involving signomial terms were solved to global optimality in the paper.

Appendix A

Below is the MILP problem (using $M = 10$, $\delta_1 = 0.01$ and $\delta_2 = 0.001$) to obtain the power transformations in the example in Sect. 6.1.

```
Minimize BB(0,0) + BB(1,1)
 + 0.01 b(0,0) + 0.01 b(0,1) + 0.01 b(1,0) + 0.01 b(1,1) + 0.001
delta(0,0) + 0.001 delta(0,1) + 0.001 delta(1,0) + 0.001 delta(1,1)
Subject to
 Eq9(0): 0.5 Q(0,0) + 2 Q(0,1) <= 1
 Eq8_0(0): b(0,0) + b(1,0) - 2 BB(0,0) <= 0
 Eq8_1(0,0): Q(0,0) - delta(0,0) + 10 beta(0,0) <= 11
 Eq8_2(0,0): -Q(0,0) - delta(0,0) + 10 beta(0,0) <= 9
 Eq8_3(0,0): Q(0,0) - delta(0,0) - 10 beta(0,0) <= -1
 Eq8_4(0,0): -Q(0,0) - delta(0,0) - 10 beta(0,0) <= 1
 Eq8_5(0,0): -Q(0,0) + 10 beta(0,0) <= 10
 Eq8_6(0,0): Q(0,0) - 10 beta(0,0) <= 0
 Eq10_1(0,0): Q(0,0) + b(0,0) >= 1
 Eq10_2(0,0): Q(0,0) + 0.1 b(0,0) <= 1
 Eq8_1(0,1): Q(0,1) - delta(0,1) + 10 beta(0,1) <= 11
 Eq8_2(0,1): -Q(0,1) - delta(0,1) + 10 beta(0,1) <= 9
 Eq8_3(0,1): Q(0,1) - delta(0,1) - 10 beta(0,1) <= -1
 Eq8_4(0,1): -Q(0,1) - delta(0,1) - 10 beta(0,1) <= 1
 Eq8_5(0,1): -Q(0,1) + 10 beta(0,1) <= 10
 Eq8_6(0,1): Q(0,1) - 10 beta(0,1) <= 0
 Eq10_1(0,1): Q(0,1) + b(0,1) >= 1
 Eq10_2(0,1): Q(0,1) + 0.1 b(0,1) <= 1
 Eq12(1): alpha(1,0) + alpha(1,1) <= 1
 Eq8_0(1): b(0,1) + b(1,1) - 2 BB(1,1) <= 0
 Eq8_1(1,0): Q(1,0) - delta(1,0) + 10 beta(1,0) <= 11
 Eq8_2(1,0): -Q(1,0) - delta(1,0) + 10 beta(1,0) <= 9
 Eq8_3(1,0): Q(1,0) - delta(1,0) - 10 beta(1,0) <= -1
 Eq8_4(1,0): -Q(1,0) - delta(1,0) - 10 beta(1,0) <= 1
 Eq8_5(1,0): -Q(1,0) + 10 beta(1,0) <= 10
 Eq8_6(1,0): Q(1,0) - 10 beta(1,0) <= 0
 Eq13_1(1,0): Q(1,0) - 10.1 alpha(1,0) <= - 0.1
 Eq13_2(1,0): Q(1,0) - 11 alpha(1,0) >= -10
 Eq14_1(1,0): b(1,0) + alpha(1,0) >= 1
 Eq14_2(1,0): b(1,0) - 0.1 Q(1,0) >= -0.1
 Eq14_3(1,0): b(1,0) + 10 alpha(1,0) - 0.9 Q(1,0) <= 10
 Eq8_1(1,1): Q(1,1) - delta(1,1) + 10 beta(1,1) <= 11
 Eq8_2(1,1): -Q(1,1) - delta(1,1) + 10 beta(1,1) <= 9
 Eq8_3(1,1): Q(1,1) - delta(1,1) - 10 beta(1,1) <= -1
 Eq8_4(1,1): -Q(1,1) - delta(1,1) - 10 beta(1,1) <= 1
 Eq8_5(1,1): -Q(1,1) + 10 beta(1,1) <= 10
 Eq8_6(1,1): Q(1,1) - 10 beta(1,1) <= 0
 Eq13_1(1,1): Q(1,1) - 10.1 alpha(1,1) <= - 0.1
 Eq13_2(1,1): Q(1,1) - 11 alpha(1,1) >= -10
 Eq14_1(1,1): b(1,1) + alpha(1,1) >= 1
 Eq14_2(1,1): b(1,1) - 0.1 Q(1,1) >= -0.1
 Eq14_3(1,1): b(1,1) + 10 alpha(1,1) - 0.9 Q(1,1) <= 10
 Eq16(1): 1.5 Q(1,0) + 1.5 Q(1,1) - 10 alpha(1,0) - 10 alpha(1,1) >= -9
Bounds
 0.1 <= Q(0,0) <= 10
 0.1 <= Q(0,1) <= 10
 -10 <= Q(1,0) <= 10
 -10 <= Q(1,1) <= 10
Binaries
 BB(0,0) BB(1,1) b(0,0) b(0,1) b(1,0) b(1,1)
 beta(1,0) beta(1,1) beta(0,1) beta(0,0) alpha(1,0) alpha(1,1)
End
```

Solving the MILP problem gives the minimum as 1.0214 with the following non-zero values of the variables:

```
BB(1,1)=1; b(0,1)=1; b(1,1)=1; beta(0,0)=1; beta(0,1)=1;
beta(1,0)=1; alpha(1,0)=1; delta(1,0)=0.75; delta(1,1)=0.66667;
Q(0,0)=1; Q(0,1)=0.25; Q(1,0)=1; Q(1,1)=-0.33333.
```

References

1. Beale, E.M.L., Forrest, J.J.H.: Global optimization using special ordered sets. Math. Program. **10**, 52–69 (1976)
2. Björk, K.-M.: A global optimization method with some heat exchanger network applications, Ph.D. thesis, Åbo Akademi University (2002)
3. Björk, K.-M., Lindberg, P.O., Westerlund, T.: Some convexifications in global optimization of problems containing signomial terms. Comp. Chem. Eng. **27**, 669–679 (2003)
4. Lundell, A.: Optimization techniques in global optimization. Master's thesis, Åbo Akademi University (2007)
5. Maranas, C.D., Floudas, C.A.: Finding all solutions of nonlinearly constrained systems of equations. J. Global Optim. **7**, 143–182 (1995)
6. Maranas, C.D., Floudas, C.A.: Global optimization in generalized geometric programming. Comp. Chem. Eng. **21**, 351–370 (1997)
7. Pörn, R.: Mixed integer non-linear programming: convexification techniques and algorithm development, Ph.D. thesis, Åbo Akademi University (2000)
8. Rijckaert, M.J., Martens, X.M.: Comparison of generalized geometric programming algorithms. J. Optim.Theory Appl. **26**(2), 205–242 (1978)
9. Westerlund, T.: Some transformation techniques in global optimization. Global Optimization: From Theory to Implementation. In: Liberti, L., Maculan, N. (eds.), pp. 47–74 Springer (2005)
10. Westerlund, T., Pörn, R.: Solving pseudo-convex mixed-integer problems by cutting plane techniques. Optim. Eng. **3**, 253–280 (2002)
11. Westerlund, T., Westerlund, J.: GGPECP—an algorithm for solving non-convex MINLP problems by cutting plane and transformation techniques. Chem. Eng. Trans. **3**, 1045–1050 (2003)